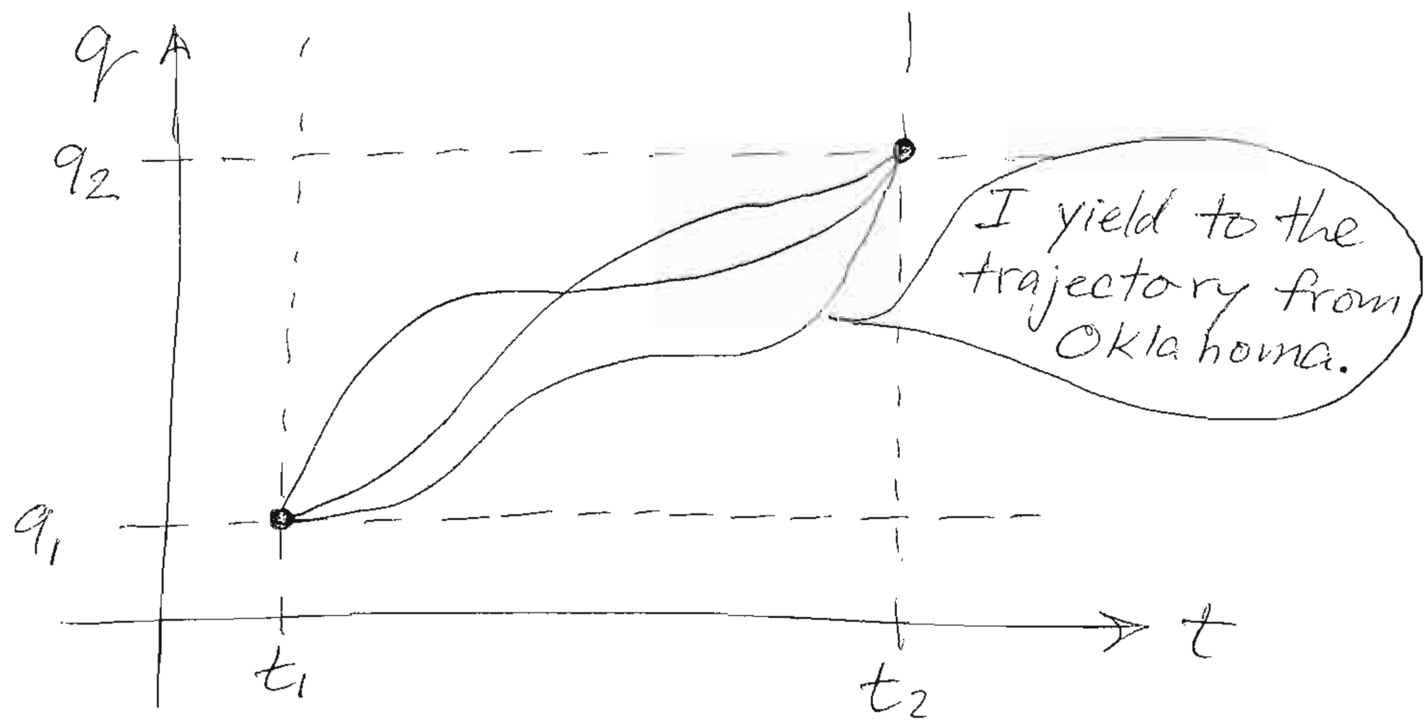
Classical Mechanics as an Approximation to Quantum Mechanics ("democracy in action")

Up to now the Lagrangian formulation of mechanics has been put forward as the method of choice for complicated problems. Not having to deal with vectors, and having the freedom to choose (generalized) coordinates, makes the Lagrangian approach superior to the Newtonian approach from the point of view of methodology. But there is another, more fundamental reason to favor the Lagrangian approach: it is closer to the truth!

The true laws of physics (at least as we understand them today) are expressed in the language of quantum mechanics. Classical mechanics emerges as an approximation of quantum mechanical behavior. The significance of the Lagrangian approach is that the nature of this approximation emerges in a very direct way.

It turns out that Q.M. can also be formulated in terms of a Lagrangian and the same action used in Hamilton's principle. There is really only one difference: rather than consider just extremal trajectories, Q.M. uses in an essential way <u>all conceivable trajectories</u> connecting the initial and final points. Interestingly, Q.M. does <u>not</u> assign any special importance (e.g. weighting factors) to the extremal trajectories singled out by Hamilton's principle. In fact, Q.M. assigns "amplitudes" (= complex numbers) to trajectories in the most democratic fashion possible: each trajectory is given a complex phase ("phasor") $e^{i\phi}$:



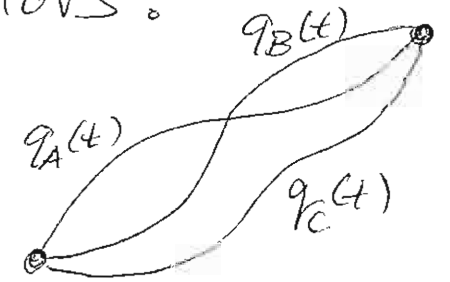
For any trajectory $q(t)$:

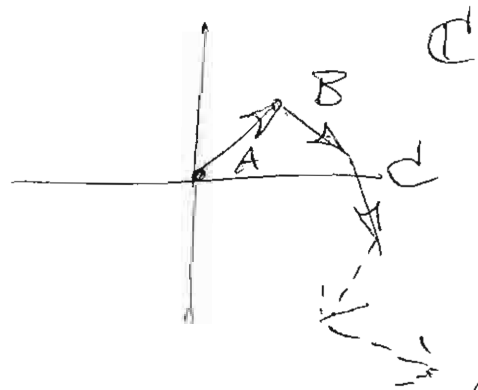
$$\phi[q(t)] = \frac{S[q(t)]}{\hbar} = \frac{1}{\hbar}\int_{t_1}^{t_2} L(q,\dot{q},t)\,dt$$

In Q.M. the net "propagation" between q_1 at $t=t_1$ and q_2 at $t=t_2$ is expressed as a single amplitude (= complex number) given by the sum of the phasors for <u>every</u> trajectory:

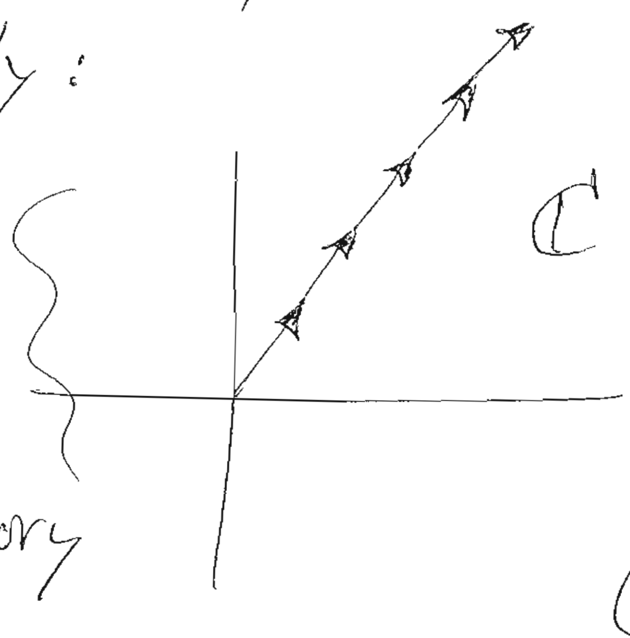
$$\mathrm{Amp}(1 \to 2) = \sum_{\substack{\text{trajectories} \\ q(t)}} e^{i\phi[q(t)]}$$

In the complex plane we represent phasors by a vector of unit magnitude. The quantum amplitude is the sum of (infinitely) many such vectors:





If the action S varies wildly (rapidly) on the scale of $\hbar$ from one trajectory to the next, then the phasors in the complex plane will be performing a kind of <u>random walk</u>. This would make us believe the net amplitude ends up being relatively small. But if we recall Hamilton's principle we realize there will always be a subset of trajectories (still an infinite number) whose phasors line up almost perfectly:



The trajectories close to the extremal trajectory (in the sense of Hamilton's principle) are special because their phase angles are very nearly equal. Perturbations of a non-extremal trajectory will produce a much larger change in S and therefore ϕ. (Recall: by construction the linear (lowest order) variation of S vanishes when we perturb the extremal trajectory.)

The near parallelism of phasors for trajectories near the extremal one is the same effect as occurs in optics called "constructive interference".

The phasor sum will be large and overwhelm the random walk contribution from "garbage" trajectories (ones far from the extremal one). Thus a "pretty good" approximation of the Q.M. sum over all trajectories is to limit the sum ~~over~~ to only those trajectories near the extremal one. Sometimes classical mechanics is said to be the $\hbar \to 0$ limit (not $\hbar = 0$) of Q.M.

Since
$$\phi[q(t)] = \frac{S[q(t)]}{\hbar},$$
we see that in order to keep the phasors nearly lined up in the limit $\hbar \to 0$, we need to make the

perturbations away from the extremal trajectory ever smaller. But that is, more or less, how we think about "classical mechanics": a single, unique trajectory.

Terms: What we've called the "extremal trajectory" — in the context of Hamilton's principle — is usually refered to as the "classical" trajectory. Also, it is customary to use the term "path" instead of "trajectory". Q.M. is said to be a "sum over paths".

Some milestones in the development of our understanding of the nature of mechanics:

1687 : the three universal laws of motion (I. Newton)

1772 : "analytical" mechanics (J. L. Lagrange)

1827 : extremal action (W. R. Hamilton)

1948 : sum over paths (R. P. Feynman)

2020? : even stranger concept.* (your name here)

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* not string theory, which is also a "sum over paths"